Polynomial Functions and Models

A ***polynomial function*** is a function in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where $a_n,\ a_{n-1}, \ldots,\ a_1,\ a_0$ are real numbers (coefficients) and n is a nonnegative integer. The domain of a polynomial function is the set of all real numbers. The ***degree*** of a polynomial is the largest power of x that appears. The zero polynomial $f(x) = 0$ is not assigned a degree.

Example: Determine which of the following are polynomial functions. For those that are, state the degree; for those that are not, tell why not.

a) $f(x) = 5 + 2x^2 - 8x^3$

yes, degree 3

b) $f(x) = 4 + 3\sqrt{x}$ $= 4 + 3x^{1/2}$

no, not allowed to have fractional exponents

c) $f(x) = -2x^3(x-1)^2$

$f(x) = -2x^3(x^2 - 2x + 1) = -2x^5 + 4x^4 - 2x^3$

yes, degree 5

d) $f(x) = 0$

yes, no degree

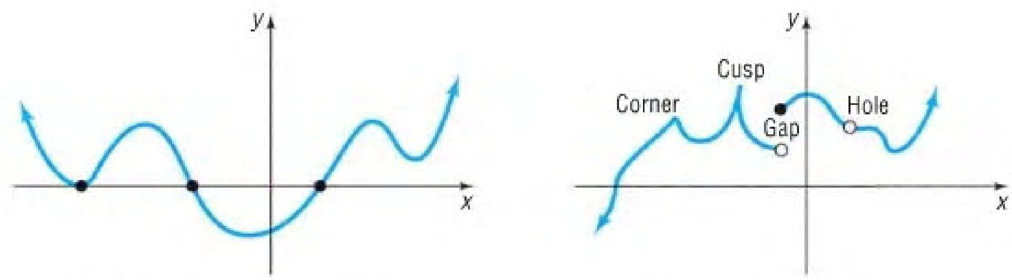
e) $f(x) = \dfrac{x^2 - 1}{x + 4}$

no – rational function, domain not $\mathbb{R}$

f) $f(x) = 9$ $\quad f(x) = 9x^0$

yes, degree 0

A polynomial function is smooth and continuous. It does not contain corners, cusps, gaps, or holes.

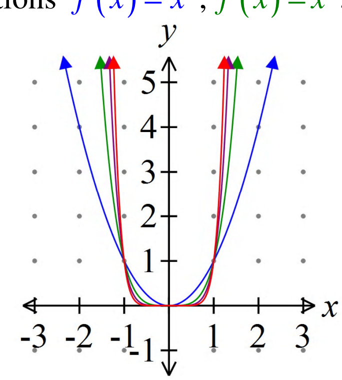


(a) Graph of a polynomial function: smooth, continuous

(b) Cannot be the graph of a polynomial function

A ***power function of degree n*** is a monomial of the form $f(x) = ax^n$, where a is a real number, $a \neq 0$, and n is an integer greater than 0.

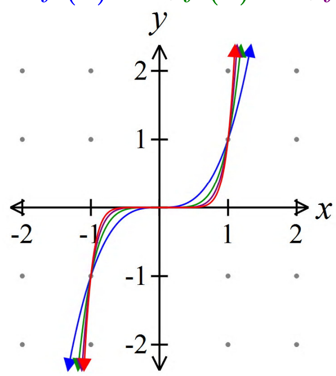
Example: Compare the even power functions $f(x) = x^2$, $f(x) = x^4$, $f(x) = x^6$, and $f(x) = x^8$.



Properties of Power Functions $f(x) = x^n$, n Is an Even Integer

1. f is an even function, so its graph is symmetric with respect to the y-axis.
2. The domain is the set of all real numbers. The range is the set of nonnegative real numbers.
3. The graph always contain the points (-1, 1), (0, 0), and (1, 1).
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

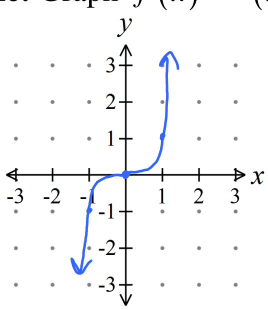
Example: Compare the odd power functions $f(x) = x^3$, $f(x) = x^5$, $f(x) = x^7$, and $f(x) = x^9$.



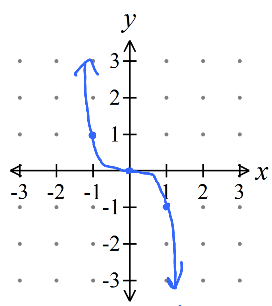
Properties of Power Functions $f(x) = x^n$, n Is an Odd Integer

1. f is an odd function, so its graph is symmetric with respect to the origin.
2. The domain and the range are the set of all real numbers.
3. The graph always contain the points (-1, -1), (0, 0), and (1, 1).
4. As the exponent n increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$; but for x near the origin, the graph tends to flatten out and lie closer to the x-axis.

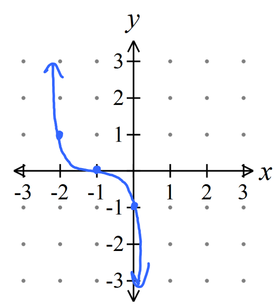
Example: Graph $f(x) = -(x+1)^5$ using transformations.



$y = x^5$

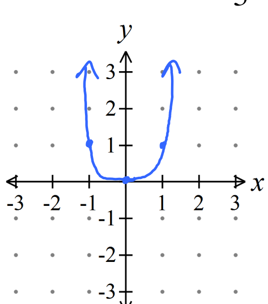


$y = -x^5$



$y = -(x+1)^5$

Example: Graph $f(x) = \frac{1}{3}x^4 - 2$ using transformations.



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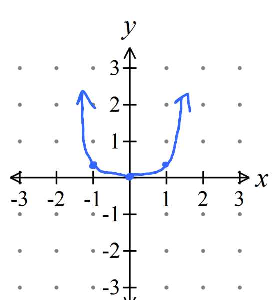
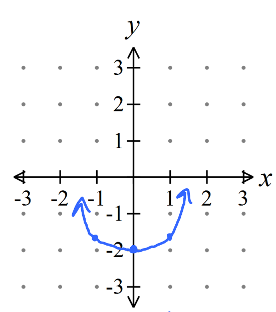


$y = \frac{1}{3}x^4$



$y = \frac{1}{3}x^4 - 2$

If f is a function and r is a real number for which $f(r)=0$, then r is called a ***real zero*** of f. The real zeros of a polynomial function are the x-intercepts of its graph, and they are found by solving the equation $f(x)=0$.

The following statements are equivalent:

1. r is a solution of $f(x)=0$.
2. r is a real zero of a polynomial function f.
3. r is an x-intercept of the graph of f.
4. $x-r$ is a factor of f.

Example:

a) Find a polynomial of degree 3 whose zeros are –2, 4, and 5.

b) Use a graphing calculator to graph the polynomial and verify the results of part a).

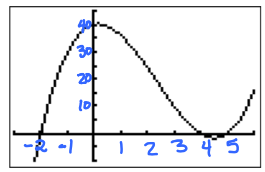
a) $f(x)=(x+2)(x-4)(x-5)$

$f(x)=(x^2-2x-8)(x-5)$

$f(x)=x^3-2x^2-8x-5x^2+10x+40$

$\boxed{f(x)=x^3-7x^2+2x+40}$

b)



If the same factor $x-r$ occurs more than once, r is called a ***repeated*, or *multiple zero* of f.** More precisely, if $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is not a factor of f, then r is called a ***zero of multiplicity m* of f.**

Example: Identify the zeros and their multiplicities for the polynomial $f(x)=3(x-4)(x+1)^3(x-2)^2$.

zero	multiplicity
4	1
-1	3
2	2

Multiplicity Rules:

If *r* is a zero of even multiplicity:

1. The sign of $f(x)$ does not change from one side to the other side of r.
2. The graph of f **touches** the x-axis at r.

If *r* is a zero of odd multiplicity:

1. the sign of $f(x)$ changes from one side to the other of r.
2. The graph of f **crosses** the x-axis at r.

Behavior Near a Zero: This will not be tested. Skip it on your homework. If you are interested, see p. 327.

Turning Points: The points at which a graph changes direction (the local maxima and local minima) are called turning points. If f is a polynomial function of degree n, then f has at most $n-1$ turning points. If the graph of a polynomial f has $n-1$ turning points, the degree of f is at least n.

End Behavior: The behavior of the graph of a function for large values of x, either positive or negative, is referred to as its end behavior. For large values of x, either positive or negative, the graph of the polynomial function $f(x)=a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ resembles the graph of the power function $y=a_nx^n$.

Example: Graph the polynomial $f(x) = x^2(x+3)$.

a) Find the x-intercepts and y-intercept of the graph of f.

b) Determine whether the graph crosses or touches the x-axis at each x-intercept.

c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

a) x-ints: $0 = x^2(x+3)$

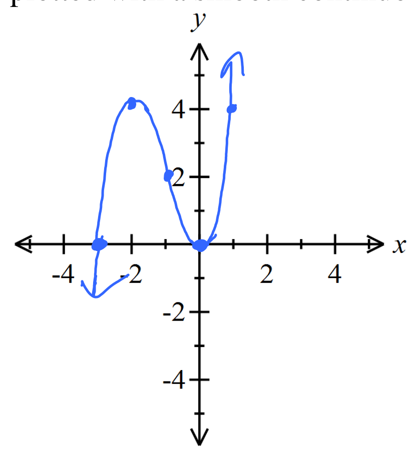
$x = 0$ (mult. 2) $\quad x = -3$ (mult. 1)

y-int: $f(0) = 0^2(0+3) = 0$

b) touches at 0,
crosses at -3

c) $y = x^2(x)$ $\quad y = x^3$ $\quad \downarrow \quad \uparrow$

d) degree 3
max turning points = 2



x	y
-2	4
-1	2
1	4
2	20

Example: Graph the polynomial $f(x) = (x+1)^2(x-3)(x-1)$.

a) Find the x-intercepts and y-intercept of the graph of f.

b) Determine whether the graph crosses or touches the x-axis at each x-intercept.

c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.

d) Determine the maximum number of turning points on the graph of f.

c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

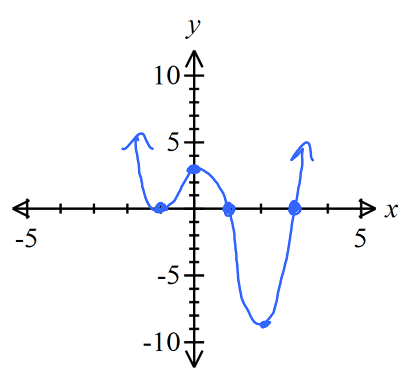
a) x-ints: -1 (mult. 2), 3 (mult. 1), 1 (mult. 1)

y-int: $f(0) = (0+1)^2(0-3)(0-1) = 3$

b) touches @ -1,
crosses @ 1 & 3

c) $y = (x^2)(x)(x)$ $\quad y = x^4$ $\quad \uparrow \quad \uparrow$

d) degree 4
max. turning pts. = 3



x	y
-2	15
2	-9
4	75

Example: Graph the polynomial $f(x) = -\frac{1}{2}x\left(x^2+1\right)(x-2)^2$.   no zeros

a) Find the x-intercepts and y-intercept of the graph of f.
b) Determine whether the graph crosses or touches the x-axis at each x-intercept.
c) End behavior: Find the power function that the graph of f resembles for large values of $|x|$.
d) Determine the maximum number of turning points on the graph of f.
c) Locate other points on the graph and connect all the points plotted with a smooth continuous curve.

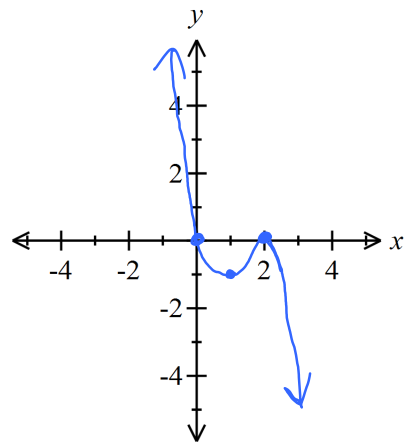
a) x-ints: 0 (mult. 1), 2 (mult. 2)

y-int: $f(0) = -\frac{1}{2}(0)(0^2+1)(0-2)^2 = 0$

b) crosses @ 0, touches @ 2

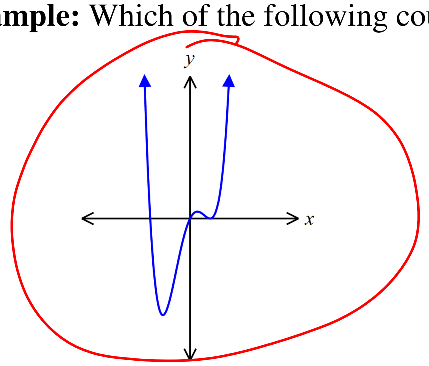
c) $y = -\frac{1}{2}x(x^2)(x^2)$ $y = -\frac{1}{2}x^5$ ↑ ↓

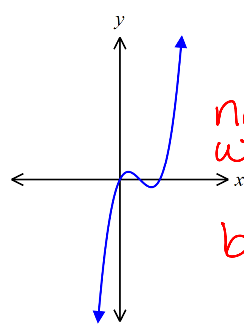
d) Degree 5
Max turning points = 4



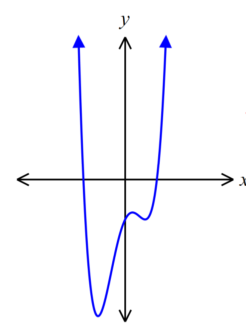
x	y
-1	9
1	-1
3	-15

Example: Which of the following could be the graph of $f(x) = x^4 - 3x^2 + 2x$?





no- wrong end behavior



no- y-int not 0.

$f(x) = x^4 - 3x^2 + 2x$

y-int: $f(0) = 0^4 - 3(0)^2 + 2(0) = 0$

end behavior: $y = x^4$ ↑ ↑

max turning points = 3